

# Clothoid Computation: A Simple and Efficient Numerical Algorithm

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**Abstract:** The clothoid (also known as Cornu spiral or Euler spiral) is a curve that is characterized by its curvature being proportional to its length. This property makes it very useful as a transition curve when designing the layout of roads and railway tracks. This paper presents an analysis of two methods for computing the clothoid: the classical method, which is based on the use of explicit formulas obtained from Taylor expansions of sine and cosine functions, and an alternative algorithm, which is based on the numerical solution of the initial value problems giving the clothoid parametrization. This alternative method is simple and efficient, and its effectiveness is shown through its application to three classical problems of horizontal road alignment. DOI: [10.1061/\(ASCE\)SU.1943-5428.0000177](https://doi.org/10.1061/(ASCE)SU.1943-5428.0000177). © 2016 American Society of Civil Engineers.

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## Introduction

In horizontal road alignments, *transition curves* are used to connect two stretches having different radii of curvature  $R_1 \neq R_2$ , so that there is a smooth change from  $R_1$  to  $R_2$ . This smooth curvature change means a gradual increase/decrease of centrifugal force experienced by the vehicle, which, in addition to avoiding the disturbance of passengers, significantly reduces the risk of accident. Moreover, transition curves for highway vertical alignments have been introduced, and analysis of these curves was recently presented (Easa and Hassan 2000a; Easa and Hassan 2000b; Kobryń 2015). Recent works suggest different types of curves as transition curves (Tari and Baykal 2005; Kobryń 2011; Bosurgi and D'Andrea 2012; Eliou and Kaliabetsos 2014; Kobryń 2014), but the transition curve that has been most commonly used in design of roads is the clothoid (Baass 1984; Kobryń 1993; Baykal et al. 1997; Dong et al. 2007).

Although the clothoid (also called Euler spiral or Cornu spiral) has been known for hundreds of years, it is currently widely used as a transition curve in the design of roads and railways and continues to be subject of interesting mathematical studies (see, for example, Korkut et al. 2008; Narayan 2014). First works on this curve date from the end of the seventeenth century. These are framed in the study of an elasticity problem, introduced in 1694 by the Swiss mathematician Jakob Bernoulli (1654–1705), and successfully solved, in the middle of the eighteenth century, by another great Swiss scientist, Leonhard Euler (1701–1783). Some years later, the curve was

rediscovered by two French physicists and used to study the diffraction of light. First, Augustin-Jean Fresnel (1788–1827) parametrized the curve in terms of integrals (the famous Fresnel integrals), and later, Marie Alfred Cornu (1841–1902) used this parameter to draw the curve accurately. Finally, its use as a transition curve for railways was introduced by the American engineer Arthur Newell Talbot (1857–1942) in the late nineteenth century [more details on the historical evolution of this curve can be found in Levien (2008)].

First, this paper presents the initial value problems giving the arc length parametrization obtained from the properties characterizing the clothoid. The next section presents a comparison of two numerical methods used to compute the clothoid: the classical method used in surveying engineering (based on the use of explicit formulas obtained from Taylor expansions of sine and cosine functions) and an alternative algorithm based on the numerical solution of the initial value problems giving the clothoid parametrization. This alternative method is simple and efficient, and its effectiveness is shown through its application to three problems of horizontal road alignment: connecting an oriented straight stretch with an arc of an oriented circle, connecting two oriented straight stretches with different directions, and connecting two oriented circles. Finally, the last section presents some brief and interesting conclusions.

## Arc Length Parametrization

The clothoid is a curve in which, at every point, the product of the radius of curvature by the arch length is constant. This property is what makes it useful as transition curve and allows for easy determination of its arc length parametrization.

Let  $C$  be a smooth plane curve of length  $L$ , and denote by  $r(s) = (x(s), y(s))$  its arc length parametrization,  $s \in [0, L]$ . Because  $r'(s) = (x'(s), y'(s))$  is a unit vector, if  $\Phi(s) \in [0, 2\pi)$  denotes the angle of the vector tangent to the curve  $C$  at the point  $r(s)$  with the positive abscissa axis ( $OX^+$ ) (see Fig. 1), it verifies that

$$\begin{cases} x'(s) = \cos \Phi(s), & s \in (0, L) \\ y'(s) = \sin \Phi(s), & s \in (0, L) \end{cases} \quad (1)$$

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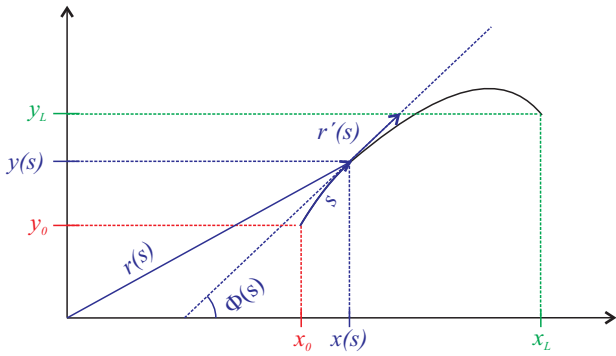


Fig. 1. Curve C, arc length parametrization

On the other hand, the curvature of C in the point  $r(s)$  is defined as

$$\kappa(s) = \|r''(s)\|$$

In this case

$$r''(s) = (x''(s), y''(s)) = \Phi'(s)(-\sin \Phi(s), \cos \Phi(s))$$

and consequently

$$\kappa(s) = |\Phi'(s)|$$

Similarly, the radius of curvature is defined as the inverse of the curvature, and therefore

$$r_c(s) = \frac{1}{|\Phi'(s)|}, s \in (0, L)$$

Because C is a clothoid, the product of the radius of curvature by the arc length is constant, and there must be  $A \in \mathbb{R}$  (called the clothoid parameter) verifying

$$sr_c(s) = A^2, s \in [0, L] \quad (2)$$

that is

$$|\Phi'(s)| = \frac{s}{A^2}, s \in (0, L)$$

Thus, if  $\Phi_0$  denotes the angle of the tangent vector to the curve C in the point  $r(0) = (x_0, y_0)$  with  $Ox^+$ , then

$$\Phi(s) = \lambda \frac{s^2}{2A^2} + \Phi_0, s \in [0, L] \quad (3)$$

where  $\lambda = 1$  if  $\Phi(s)$  is increasing (if the clothoid is traversed in the counterclockwise direction) and  $\lambda = -1$  if  $\Phi(s)$  is decreasing (if the clothoid is traversed in the clockwise direction) (see Fig. 3).

Replacing the previous expression in Eq. (1), the clothoid parametrization is given as the solution of the following initial value problems:

$$\begin{cases} x'(s) = \cos\left(\lambda \frac{s^2}{2A^2} + \Phi_0\right), & s \in (0, L), \\ x(0) = x_0 \end{cases} \quad (4)$$

$$\begin{cases} y'(s) = \sin\left(\lambda \frac{s^2}{2A^2} + \Phi_0\right), & s \in (0, L), \\ y(0) = y_0 \end{cases} \quad (5)$$

### Remark 1

In some applications, for example, to compute the clothoid linking a straight stretch with a given point of a circle arc (see Fig. 4), the initial point  $r(0) = (x_0, y_0)$  is not known, but the end point  $r(L) = (x_L, y_L)$  is known (it is the given point). In this case, the initial conditions  $x(0) = x_0$  and  $y(0) = y_0$  in Eqs. (4) and (5) have to be changed by the end conditions  $x(L) = x_L, y(L) = y_L$ , so that two final value problems are solved instead of two initial value problems.

### Remark 2

The clothoid parameter A is completely determined if the radius of curvature is known at any point of the curve (for any value of s). For example, if the point is to compute the clothoid with a radius of curvature R at the end point (when  $s = L$ ), Eq. (2) leads to  $RL = A^2$  and, consequently, the clothoid will be given by parameter  $A = \sqrt{RL}$ .

Similarly, for any value of  $A > 0$ , the expression in Eq. (2) implies

$$\lim_{s \rightarrow 0^+} r_c(s) = +\infty$$

which means that the curvature of any clothoid at the initial point  $r(0) = (x_0, y_0)$  is null. Two curves link at a point when the two curves have the same slope and the same curvature at that point. In this sense, all clothoids link at the starting point with the straight line passing through that point, with slope given by  $\Phi(0) = \Phi_0$ .

### Numerical Computation

As noted, for a given parameter A, there exist two clothoids coming out of the point  $r(0) = (x_0, y_0)$  with slope given by  $\Phi(0) = \Phi_0$ . Once it is determined which of the two clothoids to compute (once the value of  $\lambda$  is fixed), the solutions of Eqs. (4) and (5) can be used to compute it. The differential equations that govern these problems are trivial, and so

$$x(s) = \int_0^s \cos\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau + x_0, s \in [0, L] \quad (6)$$

$$y(s) = \int_0^s \sin\left(\lambda \frac{\tau^2}{2A^2} + \Phi_0\right) d\tau + y_0, s \in [0, L] \quad (7)$$

With these expressions, it should be noted that:

1. The clothoid of parameter A starting at  $r(0) = (0, 0)$ , tangent to  $Ox^+$  ( $\Phi(0) = 0$ ) and for  $s > 0$  that is above the axis  $Ox$  ( $\lambda = 1$ ), is given by

$$\hat{x}(s) = \int_0^s \cos\left(\frac{\tau^2}{2A^2}\right) d\tau, s \in [0, L] \quad (8)$$

$$\hat{y}(s) = \int_0^s \sin\left(\frac{\tau^2}{2A^2}\right) d\tau, s \in [0, L] \quad (9)$$

Any other clothoid of parameter A can be obtained by a translation [moving the origin to the point  $(x_0, y_0)$ ], a rotation (of angle  $\Phi_0$ ), and, eventually (when  $\lambda = -1$ ), a symmetry with respect to the new axis (the axis  $Ox$  translated and rotated). This property makes it necessary to compute that clothoid, call it *standard*, and then apply to it the required basic transformations. This approach is widely used in surveying engineering, and the next section presents a widespread method for the computation of that clothoid.

2. The integrals appearing in Eqs. (6) and (7) have been thoroughly studied and, for example, for the case  $\Phi_0 = 0$ , with the introduction of a new dummy variable  $\sigma = \tau/(\sqrt{\pi A})$  in Eqs. (8) and (9), these integrals are transformed into the known Fresnel integrals [see, for example, Abramowitz and Stegun (1972) and the references therein]. The direct computation of these integrals is not possible, and it is necessary to use some kind of numerical approximation. Thus, it can be more useful to employ a numerical method for directly solving the initial value problems in Eqs. (4) and (5), rather than the integrals in Eqs. (6) and (7). This approach, equivalent to the other (see Remark 3), has the advantage of, by applying the method in a previous step, allowing for more control over the model. The section discussing the alternative method presents an example of the algorithm obtained by applying the simplest numerical method to solve the problems in Eqs. (4) and (5).

### The Classical Method

As noted, any clothoid of parameter  $A$  can be obtained by basic transformations from the standard clothoid, given by Eqs. (8) and (9). To compute the integrals in Eqs. (8) and (9), the following algorithm is widely used:

1. Consider the McLaurin polynomials of a certain degree for  $\cos x$  and  $\sin x$ .
2. Replace the values of the integrands in Eqs. (8) and (9) with these polynomials evaluated at point  $\tau^2/(2A^2)$ .
3. Compute the resulting integral with the use of Barrow's rule.

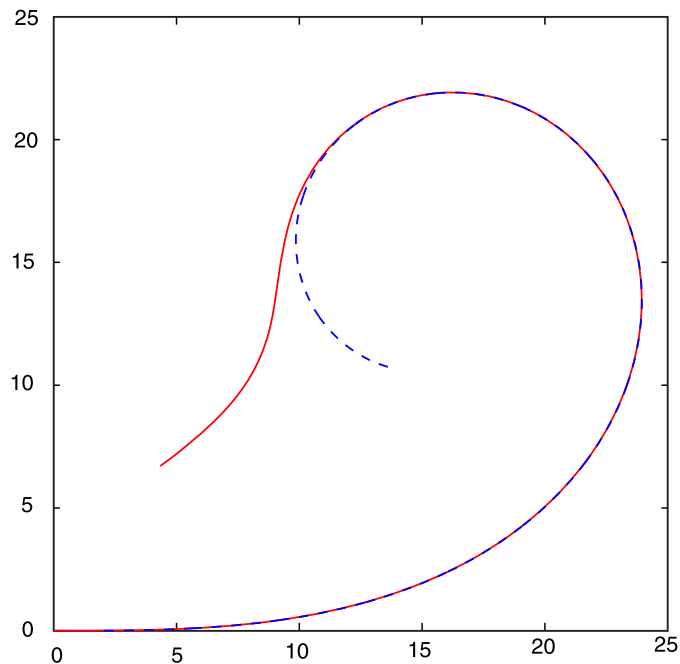
For example, if the polynomials of degree 10 for  $\cos x$  and degree 11 for  $\sin x$  are considered, the following approximations (see Abramowitz and Stegun 1972) are obtained:

$$\hat{x}(s) \approx s - \frac{s^5}{2!5(\sqrt{2A})^4} + \frac{s^9}{4!9(\sqrt{2A})^8} - \frac{s^{13}}{6!13(\sqrt{2A})^{12}} + \frac{s^{17}}{8!17(\sqrt{2A})^{16}} - \frac{s^{21}}{10!21(\sqrt{2A})^{20}}, s \in [0, L], \quad (10)$$

$$\hat{y}(s) \approx \frac{s^3}{2!3A^2} - \frac{s^7}{3!7(\sqrt{2A})^6} + \frac{s^{11}}{5!11(\sqrt{2A})^{10}} - \frac{s^{15}}{7!15(\sqrt{2A})^{14}} + \frac{s^{19}}{9!19(\sqrt{2A})^{18}} - \frac{s^{23}}{11!23(\sqrt{2A})^{22}}, s \in [0, L] \quad (11)$$

It should be noted that the McLaurin series of  $\cos x$  and  $\sin x$  are approximations of these functions in the neighborhood of zero, so that these approaches are only good for small values of  $\tau^2/(2A^2)$ . However, the dummy variable  $\tau$  takes values in the interval  $[0, s]$  and, consequently, the previous approaches can present problems for large values of  $s$ . To illustrate this fact, Fig. 2 shows the standard clothoid of parameter  $A = 17.32$  with a dashed line, and the approximation given by Eqs. (10) and (11) with a solid line. Fig. 2 shows that the approximation is very good for small values of  $s$ , but after a certain value, as expected, it becomes much less accurate.

Despite the limitations, it is essential to note that the approaches in Eqs. (10) and (11) are perfectly valid for their applications in surveying engineering. In effect, as mentioned in Remark 2, the clothoid ending with radius of curvature  $R$  corresponds with a parameter  $A = \sqrt{RL}$ . Consequently, the approximations in Eqs. (10) and (11) are good if  $s^2/(2LR)$  is not too large, and because of  $s \in [0, L]$ , this happens whenever  $L/(2R)$  is not too large. The



**Fig. 2.** Approximations of the clothoid of parameter  $A = 17.32$ , for an arc length  $L = 60$ ; the solid line was computed with the classical method, Eqs. (10)–(11), and the dashed line with the alternative method, Eq. (12)

numerical experiments show that these approximations are good enough for values of  $L/(2R)$  lower than 3. After this value, the clothoid is distorted and stops being useful as transition curve.

### An Alternative Method

An alternative way to proceed is to obtain the desired clothoid parametrization using a numerical method to solve the problems in Eqs. (4) and (5). There are many methods that can be used (see, e.g., Atkinson et al. 2009), and each one of them leads to a different algorithm. As an example, an algorithm can be obtained with use of the simple explicit Euler's method, as follows.

Let  $L > 0$ . Choose a natural number  $N \in \mathbb{N}$ , define  $\Delta s = L/N$ , and, for each  $n = 0, 1, \dots, N$ , consider  $s^n = n\Delta s$ . Euler's method is based on the approximations

$$x'(s^n) = \frac{x(s^{n+1}) - x(s^n)}{\Delta s}, \quad y'(s^n) = \frac{y(s^{n+1}) - y(s^n)}{\Delta s}$$

Thus, define  $x^0 = x_0$ ,  $y^0 = y_0$ , and for all  $n = 0, \dots, N - 1$  compute

$$\begin{aligned} x^{n+1} &= x^n + \Delta s \cos \left( \lambda \frac{(s^n)^2}{2A^2} + \Phi_0 \right), \\ y^{n+1} &= y^n + \Delta s \sin \left( \lambda \frac{(s^n)^2}{2A^2} + \Phi_0 \right) \end{aligned} \quad (12)$$

and accept the approximations  $x(s^{n+1}) \approx x^{n+1}$ ,  $y(s^{n+1}) \approx y^{n+1}$ .

### Remark 3

The method just presented is the same as that obtained if the integrals in Eqs. (6) and (7) are computed using the left rectangles

**Table 1.** Comparison of Classical Method and Alternative Method for Computing a Standard Clothoid of Parameter  $A = 17.32$

$s = s^n$	Alternative		Classical		Differences		Error estimate
	$x^n$	$y^n$	$\hat{x}(s^n)$	$\hat{y}(s^n)$	$ x^n - \hat{x}(s^n) $	$ y^n - \hat{y}(s^n) $	$\frac{(\Delta s)^3 n(n+1)}{4A^2}$
0	0	0	0	0	0	0	0
15	14.7908	1.8541	14.7904	1.8563	0.0004	0.0022	0.0023
30	23.9233	12.7493	23.9177	12.7553	0.0056	0.0060	0.0090
45	14.6790	21.7373	14.6594	21.7340	0.0196	0.0033	0.0203
55	10.1555	14.0968	9.0509	13.6903	1.1046	0.4065	0.0303
60	13.6087	10.7421	4.3401	6.7009	9.2685	4.0411	0.0360

formula. Similarly, solving the problems in Eqs. (4) and (5) with the modified Euler’s method would result in the same algorithm as computing the integrals in Eqs. (6) and (7) with the trapezoidal rule. In general, using a numerical method to solve Eqs. (4) and (5) is equivalent to approximating the integrals in Eqs. (6) and (7) with a numerical quadrature formula.

Despite its simplicity, the previous algorithm is effective, and if  $N$  is large enough ( $\Delta s$  is small enough), the values of  $x^n$  and  $y^n$  obtained are good approximations of  $x(s^n)$  and  $y(s^n)$ . In fact, the following error estimates (see Appendix) guarantee the previous affirmation:

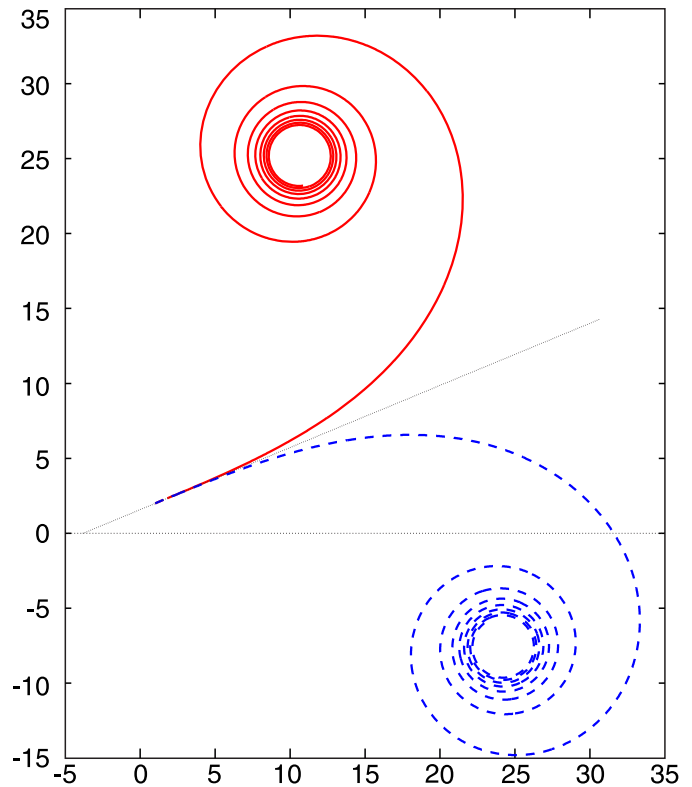
$$|x^n - x(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2},$$

$$|y^n - y(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2}$$

To illustrate the accuracy of both methods (classical and alternative), the standard clothoid of parameter  $A = 17.32$  (see Fig. 2) was computed with both algorithms. Table 1 shows the coordinates of some points obtained by both methods, the differences between them, and the error estimate for the alternative method. Table 1 shows that the error estimate was always small (the alternative method had good accuracy). For small values of  $s$ , both methods gave similar points, but for large values of  $s$ , the differences were much larger than the error estimate, showing the poor performance of the classical method for large values of  $s$  (which is also reflected in Fig. 2). The good performance of the alternative method is illustrated in Fig. 3, which shows the approximations of the two clothoids of parameter  $A = 20$  that link at point  $(x_0, y_0) = (1, 2)$  with the straight line of slope given by  $\Phi_0 = \pi/8$ . Unlike with the result obtained with the use of the method described in the previous section, this algorithm directly computes the needed clothoid (without later translations, rotations, and/or symmetries), and also its behavior is acceptable for large values of  $s$ .

### Applications

This section presents three examples in which the clothoid is used as transition curve. First, the clothoid is used to connect a straight stretch oriented (given by a free vector  $\mathbf{v}$ ) with a particular point in a given oriented circle. Second, it is used to design the layout of a road, with known principal directions (intersecting lines) and the permitted minimum radius of curvature. Finally, by combining the previous applications, the clothoid is applied to the transition between two oriented circles.

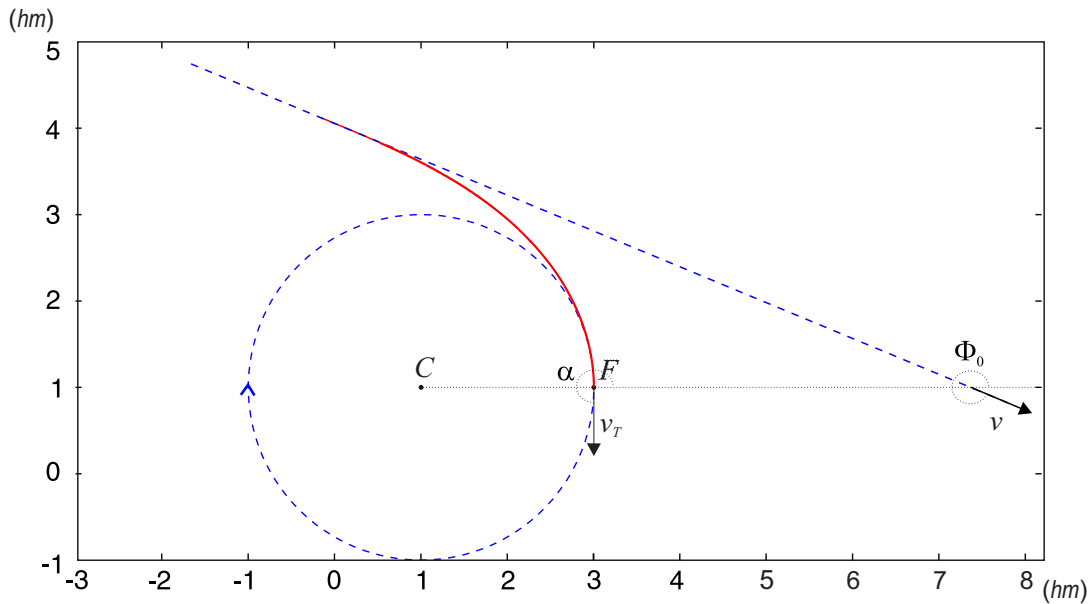


**Fig. 3.** Approximations of the two clothoids of parameter  $A = 20$  that link at point  $(x_0, y_0) = (1, 2)$  with the straight line of slope given by  $\phi_0 = \pi/8$ ; the solid line corresponds with  $\lambda = 1$  and the dashed line with  $\lambda = -1$

### Connecting an Oriented Straight Stretch with an Arc of an Oriented Circle

Consider a straight stretch whose direction and sense are determined by a known vector  $\mathbf{v}$  (see Fig. 4). The aim is to connect with a point  $F = (x_F, y_F)$  of a circle centered on  $C = (x_C, y_C)$ , to be traveled with an orientation given by the indicator  $\lambda_C$  ( $\lambda_C = -1$  clockwise,  $\lambda_C = 1$  counterclockwise). The aim is to obtain the shortest clothoid providing such connection. The following observations can be made:

1. Obviously, the sign of the clothoid ( $\lambda$ ) is determined by the circle orientation, such that  $\lambda = \lambda_C$ .
2. The end point of the clothoid must be linked to point  $F$ . To ensure that (as already proposed in Remark 1), in Eqs. (4) and (5), the initial conditions must be replaced with the final terms conditions



**Fig. 4.** Clothoid connecting a straight stretch, with direction and sense given by  $\mathbf{v} = (\cos 15\pi/8, \sin 15\pi/8)$ , with the point  $F = (3,1)$  of the circle centered on  $C = (1,1)$  and clockwise oriented

$$x(L) = x_F, \quad y(L) = y_F$$

3. When the clothoid is linked with the circle at point  $F$ , both curves must have the same slope and the same radius of curvature. This property determines the length of the clothoid, as follows:

- a. First, the radius of curvature at the link point is computed as the norm of the vector  $\mathbf{CF} = \mathbf{OF} - \mathbf{OC}$ . Thus

$$\mathbf{CF} = (x_F - x_C, y_F - y_C)$$

and

$$R = \sqrt{(x_F - x_C)^2 + (y_F - y_C)^2}$$

- b. From vector  $\mathbf{CF}$ , and taking into account the orientation of the circle, the tangent vector at point  $F$  is computed, which is given by

$$\mathbf{v}_T = \lambda_C(y_C - y_F, x_F - x_C)$$

- c. The angles of vectors  $\mathbf{v}$  and  $\mathbf{v}_T$  are denoted by  $\Phi_0 \in [0, 2\pi)$  and  $\alpha \in [0, 2\pi)$ , respectively, with  $\text{OX}^+$  (see Fig. 4).
- d. For the clothoid and circle to have the same slope at the link point, it must be verified that  $\Phi(L) = \alpha$ . Then, using Eq. (3) and taking into account that  $A = \sqrt{RL}$ , gives

$$\lambda \frac{L}{2R} + \Phi_0 = \alpha \quad (13)$$

Denoting the angle that must be rotated (*rotation angle*) by  $\tau \in [0, 2\pi)$  gives

$$\tau = \begin{cases} \lambda(\alpha - \Phi_0) & \text{if } \lambda(\alpha - \Phi_0) \geq 0, \\ 2\pi + \lambda(\alpha - \Phi_0) & \text{if } \lambda(\alpha - \Phi_0) < 0 \end{cases}$$

and, from Eq. (13),  $L = 2R\tau$  is obtained.

Taking these calculations into account, the clothoid is now completely determined; the end point, the minimum radius  $R$ , its length

$L$  (and consequently its parameter  $A = \sqrt{RL}$ ), and the values of  $\lambda$  and  $\Phi_0$  are already known. The alternative method described in previous section gives the clothoid as follows:

Define  $x^N = x_F, y^N = y_F$ ; for all  $n = N - 1, \dots, 0$  compute

$$\begin{aligned} x^n &= x^{n+1} - \Delta s \cos \left( \lambda \frac{(s^n)^2}{2A^2} + \Phi_0 \right), \\ y^n &= y^{n+1} - \Delta s \sin \left( \lambda \frac{(s^n)^2}{2A^2} + \Phi_0 \right) \end{aligned} \quad (14)$$

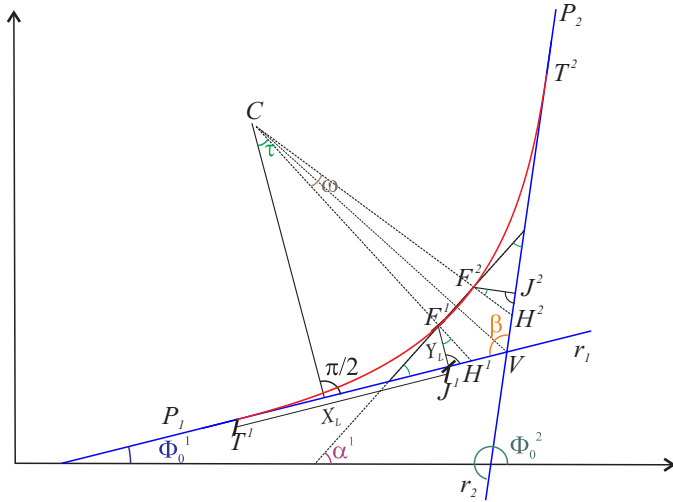
and approach  $r(s^n) = (x(s^n), y(s^n)) \approx (x^n, y^n)$ .

As an example, Fig. 4 shows the clothoid connecting a straight stretch, with direction and sense given by  $\mathbf{v} = (\cos(15\pi/8), \sin(15\pi/8))$ , with the point  $F = (3,1)$  of the circle centered on  $C = (1,1)$  and clockwise oriented. As indicated earlier, it should be noted that with these data the clothoid is uniquely determined. The straight line of linkage (the line linking with the clothoid) must not be a data point, but it is obtained, a posteriori, from the *director vector*  $\mathbf{v}$ , once the initial point of the clothoid,  $r(0) \approx (x^0, y^0)$ , is computed.

### Connecting Two Oriented Straight Stretches

Suppose that one wants to connect two straight stretches intersecting at a point  $V$ . To set directions (and senses), one should assume two known points,  $P_1$  and  $P_2$  (see Fig. 5). The aim is to connect the half-line from  $P_1$  to  $V$  ( $r_1$ ) with the other one from  $V$  to  $P_2$  ( $r_2$ ). Once the minimum radius of curvature  $R$  that can have the road in this stretch is known, the aim is to find two clothoids (one that begins tangent to  $r_1$ , with the sense of vector  $\mathbf{P}_1\mathbf{V} = \mathbf{OV} - \mathbf{OP}_1$ , and another that begins tangent to  $r_2$ , with the sense of vector  $\mathbf{P}_2\mathbf{V} = \mathbf{OV} - \mathbf{OP}_2$ ). These clothoids must be connected by an arc of a circle of radius  $R$  and given angle  $\omega \geq 0$  (see Fig. 5). Next, the algorithm to compute these clothoids is determined, and thereby the layout of the road is designed (hereafter, the superscript 1 refers the clothoid that departs from  $r_1$ , and 2 refers the one that departs from  $r_2$ ; matching parameters in both clothoids do not carry a superscript), as follows:

1. From points  $P_1, P_2$  and  $V$ , compute the following:
  - a. The angle  $\Phi_0^1 \in (0, 2\pi)$  between  $\mathbf{P}_1\mathbf{V}$  and  $\text{OX}^+$ ;



**Fig. 5.** Scheme of the curve clothoid–circle–clothoid, used to connect two straight stretches

- b. The angle  $\Phi_0^2 \in (0, 2\pi)$  between  $\mathbf{P}_2\mathbf{V}$  and  $\text{OX}^+$ ;  
c. The angle  $\beta \in (0, \pi)$  between the two straight lines; and  
d. The signs of clothoids  $\lambda^1$  and  $\lambda^2$  (obviously  $\lambda^2 = -\lambda^1$ ).
2. From angle  $\beta$ , compute the angle  $\tau \in (0, \pi/2)$  (the angle that forms the tangent vector to the curve at point  $F^1$  with the vector  $\mathbf{P}_1\mathbf{V}$ ). Fig. 5 shows that  $\tau + \omega/2$  is the complementary of  $\beta/2$ , and thus

$$\tau = \frac{\pi - (\omega + \beta)}{2}$$

3. From radius  $R$  and angle  $\tau$  (Fig. 5 shows that  $\tau = \alpha^1 - \Phi_0^1$  is the rotation angle), compute the length of the clothoids. As given in the previous example,  $L = 2R\tau$ , and the parameter of the clothoids is  $A = \sqrt{RL}$ .
4. From the parameter  $A$  of the clothoids, compute the points of tangency at  $r_1$  and  $r_2$  (points  $T^1$  and  $T^2$ , respectively), proceeding as follows:  
a. Compute the end point  $(X_L, Y_L)$  of the standard clothoid of parameter  $A$ . For example, using the alternative method gives

$$X_L = \Delta s \sum_{k=0}^{N-1} \left( \cos \left( \frac{(s^k)^2}{2A^2} \right) \right),$$

$$Y_L = \Delta s \sum_{k=0}^{N-1} \left( \sin \left( \frac{(s^k)^2}{2A^2} \right) \right)$$

- b. From  $X_L$  and  $Y_L$ , compute the proper distances of the clothoid (see Fig. 5):

$$\text{dist}(T^1, J^1) = \text{dist}(T^2, J^2) = X_L,$$

$$\text{dist}(J^1, H^1) = \text{dist}(J^2, H^2) = Y_L \tan \tau,$$

$$\text{dist}(H^1, V) = \text{dist}(H^2, V) = \left( R + \frac{Y_L}{\cos \tau} \right) \left( \frac{\sin(\omega/2)}{\sin(\beta/2)} \right),$$

$$\begin{aligned} \text{dist}(T^1, V) = \text{dist}(T^2, V) = \text{dist}(T^1, J^1) + \text{dist}(J^1, H^1) \\ + \text{dist}(H^1, V). \end{aligned}$$

- c. From the previous distances, compute the points of tangency  $T_1$  and  $T_2$  given by

$$\mathbf{OT}^1 = \mathbf{OV} - \text{dist}(T^1, V) \frac{\mathbf{P}_1\mathbf{V}}{\|\mathbf{P}_1\mathbf{V}\|},$$

$$\mathbf{OT}^2 = \mathbf{OV} - \text{dist}(T^2, V) \frac{\mathbf{P}_2\mathbf{V}}{\|\mathbf{P}_2\mathbf{V}\|}$$

5. Taking point  $T^1$  (respectively  $T^2$ ) as the initial point, use the alternative method given by Eq. (12) to compute the clothoid of parameter  $A$ , initial angle  $\Phi_0^1$  (respectively  $\Phi_0^2$ ), and sign  $\lambda^1$  (respectively  $\lambda^2$ ).

The previous algorithm is described for a fixed system of reference and then can be used to connect several different straight stretches with curves of different radius. This can be very useful in the design (layout) of a road. As an example, Fig. 6 shows the path of a road that links three straight stretches, the first two with a clothoid–clothoid curve of minimum radius  $R = 1.5$  hm and the next two with a clothoid–circle–clothoid curve of radius  $R = 1$  hm and angle  $\omega = \pi/8$ . To use the algorithm, the straight lines are determined by the following points:  $P_1 = (0, 2)$ ,  $V_1 = (1, 1)$ ,  $P_2 = V_2 = (15, -1)$ , and  $P_3 = (12, 3)$ .

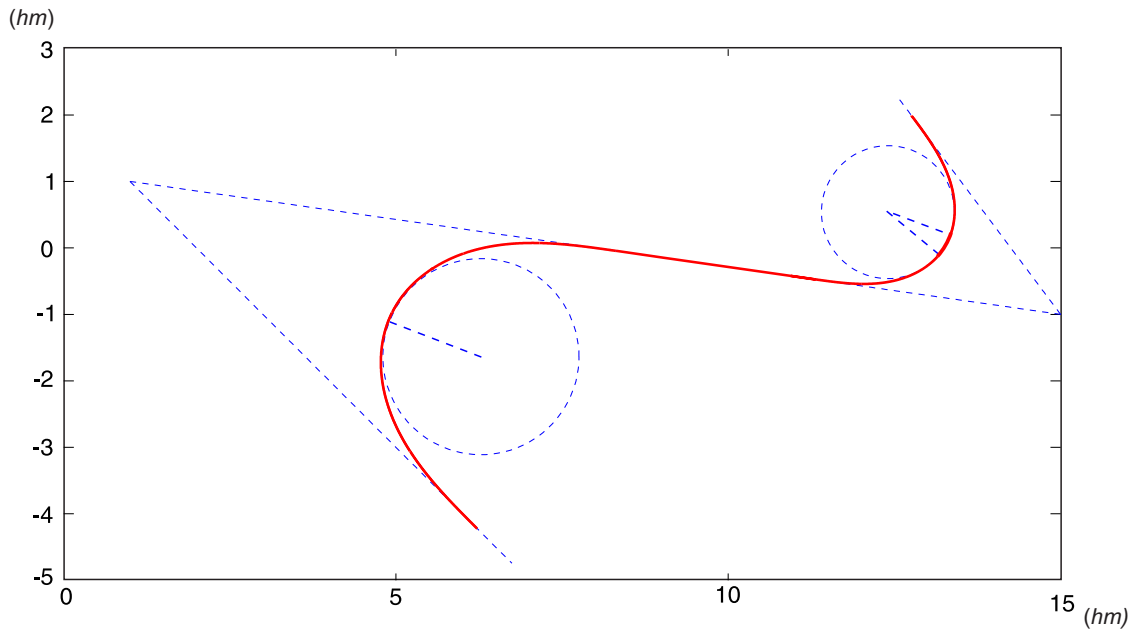
### Connecting Two Oriented Circles

Combining previous applications allows for the easy connection of two oriented circles. In this case, consider two circles centered on  $C_1 = (x_{C1}, y_{C1})$  and  $C_2 = (x_{C2}, y_{C2})$ , to be traveled with orientations given, respectively, by indicators  $\lambda_{C1}$  and  $\lambda_{C2}$ . The aim is to connect a point on the first circle  $F_1 = (x_{F1}, y_{F1})$  with another point on the second circle  $F_2 = (x_{F2}, y_{F2})$ , with the use of a transition curve that combines the clothoids (possibly connected by straight stretches and/or arc circles). The process is as follows:

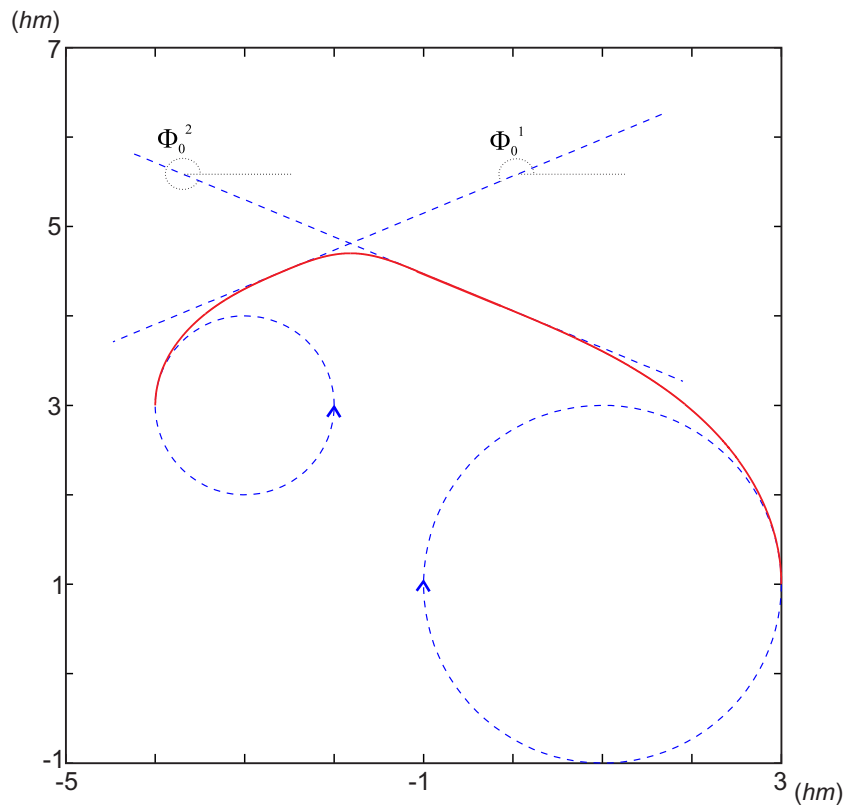
- *Step 1:* Choose two straight directions (angles  $\Phi_0^1, \Phi_0^2 \in [0, 2\pi)$ ) determining the slope of the clothoids at the initial points (Fig. 7). Choose the minimum radius permitted,  $R > 0$ , and the angle  $\omega \geq 0$  determining the arc circle used to combine the clothoids (obviously, if  $\omega = 0$ , the arc circle is not used). All of these parameters should be chosen by the engineers according to their own preferences or needs (e.g., to avoid obstacles, minimize the road length, maximize safety).
- *Step 2:* As described previously, compute the clothoids connecting the straight stretches with slopes given by  $\Phi_0^1$  and  $\Phi_0^2$ , with the oriented circles at points  $F_1$  and  $F_2$ .
- *Step 3:* As described previously, compute the transition curve (clothoid–circle–clothoid or clothoid–clothoid) connecting the two oriented straight stretches completely determined in Step 2.

As an example, Fig. 7 shows the transition curve obtained with the previous method to connect, at points  $F_1 = (-4, 3)$  and  $F_2 = (3, 1)$ , the circles centered, respectively, in  $C_1 = (-3, 3)$  and  $C_2 = (1, 1)$ . Assume that the circles have opposite orientations ( $\lambda_{C1} = 1$ ,  $\lambda_{C2} = -1$ ), and choose  $\Phi_0^1 = 9/8\pi$ ,  $\Phi_0^2 = 15/8\pi$ ,  $R = 1$  hm,  $\omega = 0$ .

Note that with  $\Phi_0^2 = \Phi_0^1 + \pi$ , the previous method can be used to connect two circles covered along the same direction (*egg-shaped transition*), and also covered along the opposite direction (*reversing circular curves*) (the same sign of  $\lambda_{C1}$  and  $\lambda_{C2}$  corresponds to reversing circular curves, and the opposite to egg-shaped transition). Fig. 8 shows both connections for two circles with radii 1 hm and 2.1 hm. In this example,  $\Phi_0^1 = 7/8\pi$  was used for the egg-shaped transition [Fig. 8(a)] and  $\Phi_0^1 = 11/8\pi$  for the reversing circular curves [Fig. 8(b)].



**Fig. 6.** Layout of a road connecting three straight stretches determined by the points  $P_1 = (0,2)$ ,  $V_1 = (1,1)$ ,  $P_2 = V_2 = (15,-1)$ , and  $P_3 = (12,3)$

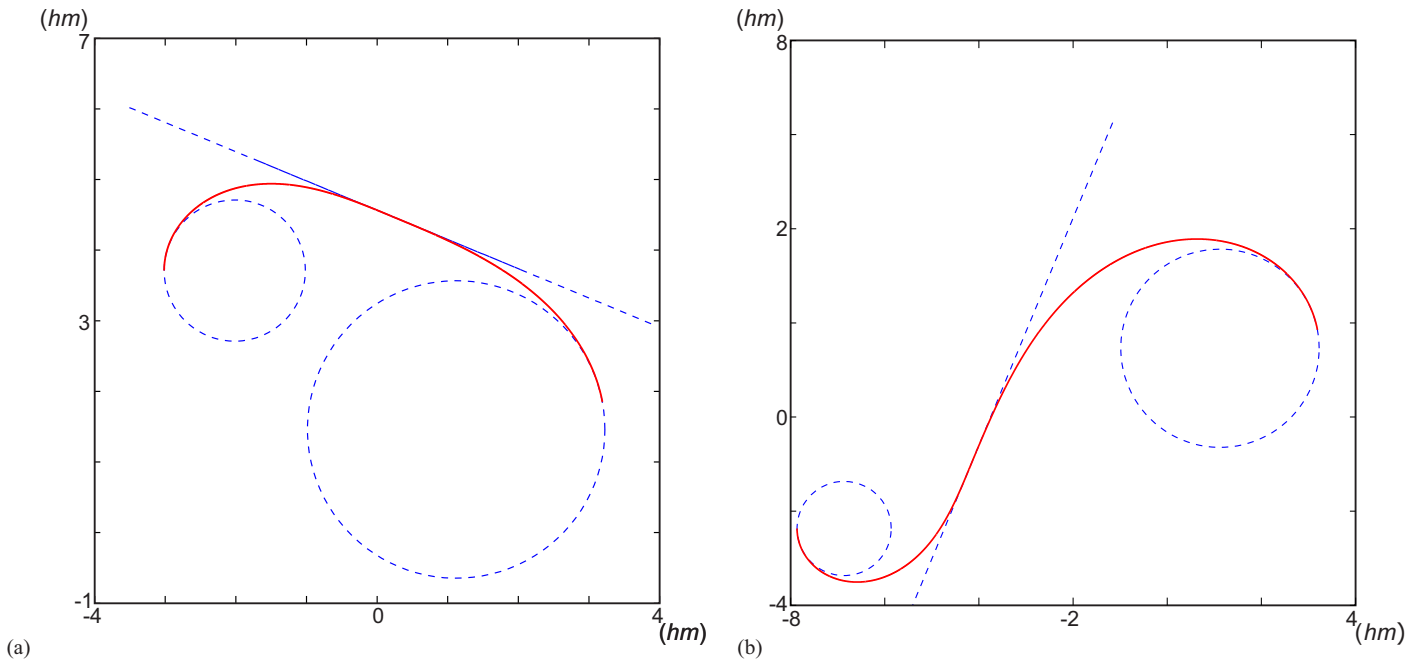


**Fig. 7.** Transition curve connecting two given points of two oriented circles

## Conclusions

This paper compared two different methods for computing clothoids: a classical method based on the use of explicit formulas obtained from Taylor expansions, and an alternative method based on the numerical solution of the initial value problems appearing in

their initial parametrization. This second method does not use complex formulas, and therefore it is conceptually very simple and easy to apply. It also provides good approximations, even when the classical method fails (when the ratio between length and radius of curvature is large), and needs no rotations, translations, or subsequent symmetries. It can be easily included in any model (computer



**Fig. 8.** (a) Egg-shaped transition; (b) transition between two reversing circular curves

application) for horizontal road design, and thus it can be a useful tool for optimizing the horizontal alignment of roads (see [Mondal et al. 2015](#)).

### Appendix. Error Estimates

This appendix details the proof of the following result, which gives useful error estimates for the alternative method.

#### Theorem 1

For all  $n = 0, 1, \dots$ , the approximations given by Eq. (12) verify

$$|x^n - x(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2}, \quad (15a)$$

$$|y^n - y(s^n)| \leq \frac{(\Delta s)^3 n(n+1)}{4A^2} \quad (15b)$$

and consequently

$$|x^n - x(s^n)| \leq \frac{L}{4R} \Delta s + \frac{1}{4R} (\Delta s)^2, \quad (16a)$$

$$|y^n - y(s^n)| \leq \frac{L}{4R} \Delta s + \frac{1}{4R} (\Delta s)^2 \quad (16b)$$

This result can be proved with the following error estimate for the left rectangles formula (see [Atkinson et al. 2009](#)).

#### Lemma 1

If  $f(x)$  is a continuously differentiable function in  $[a, b]$ , then

$$\left| \int_a^b f(x) dx - f(a)(b-a) \right| \leq M \frac{(b-a)^2}{2}$$

where  $M = \max_{x \in [a,b]} |f'(x)|$ .

With a direct application of this lemma, the following result is obtained.

#### Corollary 1

$$\left| \int_{s^k}^{s^{k+1}} \cos \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right) d\tau - \Delta s \cos \left( \lambda \frac{(s^k)^2}{2A^2} + \Phi_0 \right) \right| \leq \frac{(k+1)(\Delta s)^3}{2A^2}, \quad (17a)$$

$$\left| \int_{s^k}^{s^{k+1}} \sin \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right) d\tau - \Delta s \sin \left( \lambda \frac{(s^k)^2}{2A^2} + \Phi_0 \right) \right| \leq \frac{(k+1)(\Delta s)^3}{2A^2} \quad (17b)$$

#### Proof

Taking  $f(\tau) = \cos \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right)$  in  $[s^k, s^{k+1}]$ , results in  $|f'(\tau)| \leq \tau/A^2$ , and consequently

$$M = \max_{\tau \in [s^k, s^{k+1}]} |f'(\tau)| \leq \frac{s^{k+1}}{A^2} = \frac{(k+1)\Delta s}{A^2}.$$

The inequality in Eq. (17a) is now obtained as a direct application of the previous lemma, and Eq. (17b) is obtained in a similar way.

Now, Theorem 1 can be proved.

#### Proof of Theorem 1

Eqs. (15a) and (16a) are obtained first, and the inequalities in Eqs. (15b) and (16b) can be obtained in a similar way.



The first equality of Eq. (12) is equivalent to

$$x^{n+1} = x_0 + \sum_{k=0}^n \Delta s \cos \left( \lambda \frac{(s^k)^2}{2A^2} + \Phi_0 \right) \quad (18)$$

Eqs. (6) and (8) result in

$$|x^n - x(s^n)| = \left| \sum_{k=0}^{n-1} \Delta s \cos \left( \lambda \frac{(s^k)^2}{2A^2} + \Phi_0 \right) - \int_0^{s^n} \cos \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right) d\tau \right|$$

Obviously

$$\int_0^{s^n} \cos \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right) d\tau = \sum_{k=0}^{n-1} \int_{s^k}^{s^{k+1}} \cos \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right) d\tau$$

Thus, Eq. (17a) gives

$$|x^n - x(s^n)| \leq \sum_{k=0}^{n-1} \left| \int_{s^k}^{s^{k+1}} \cos \left( \lambda \frac{\tau^2}{2A^2} + \Phi_0 \right) d\tau - \Delta s \cos \left( \lambda \frac{(s^k)^2}{2A^2} + \Phi_0 \right) \right|$$

$$\leq \sum_{k=0}^{n-1} \frac{(k+1)(\Delta s)^3}{2A^2} = \frac{(\Delta s)^3}{2A^2} \sum_{k=1}^n k = \frac{(\Delta s)^3 n(n+1)}{4A^2}$$

Finally, taking into account that  $\Delta s = L/N$ ,  $A = \sqrt{RL}$  and  $n \leq N$ , the result is

$$|x^n - x(s^n)| \leq \frac{(\Delta s)^3 N(N+1)}{4RL} = \frac{(\Delta s)^2}{4R} \left( \frac{L}{\Delta s} + 1 \right)$$

$$= \frac{L}{4R} \Delta s + \frac{1}{4R} (\Delta s)^2$$

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